EXACT LOCAL RECONSTRUCTION ALGORITHMS FOR SIGNALS WITH FINITE RATE OF INNOVATION

Pier Luigi Dragotti

Electrical and Electronic Engineering
Imperial College London
Exhibition Road, London SW7-2AZ, UK
p.dragotti@imperial.ac.uk

Martin Vetterli[1] *and Thierry Blu*[2]

[1] LCAV, [2] BIG
École Polytechnique Fédérale de Lausanne
CH - 1015 Lausanne, Switzerland.
{martin.vetterli,thierry.blu}@epfl.ch

ABSTRACT

Consider the problem of sampling signals which are not bandlimited, but still have a finite number of degrees of freedom per unit of time, such as, for example, piecewise polynomial or piecewise sinusoidal signals, and call the number of degrees of freedom per unit of time the rate of innovation. Classical sampling theory does not enable a perfect reconstruction of such signals since they are not bandlimited.

In this paper, we show that many signals with finite rate of innovation can be sampled and perfectly reconstructed using kernels of compact support and a local reconstruction algorithm. The class of kernels that we can use is very rich and includes functions satisfying Strang-Fix conditions, Exponential Splines and functions with rational Fourier transforms. Extension of such results to the 2-dimensional case are also discussed and an application to image super-resolution is presented.

Index Terms— Sampling, wavelet theory, moments, spectral analysis, super resolution.

1. INTRODUCTION

Sampling theory plays a central role in modern signal processing and communications, and has experienced a recent revival thanks, in part, to the recent advances in wavelet theory [9]. In the typical sampling setup depicted in Figure 1, the original continuous-time signal $x(t)$ is filtered before being (uniformly) sampled with sampling period T. The filtering may be a design choice or, as it is usually the case, may be due to the acquisition device. If we denote with $y(t) = h(t) * x(t)$ the filtered version of $x(t)$, the samples y_n are given by

$$y_n = \langle x(t), \varphi(t/T - n) \rangle = \int_{-\infty}^{\infty} x(t)\varphi(t/T - n)dt$$

where the sampling kernel $\varphi(t)$ is the scaled and time-reversed version of $h(t)$.

The key problem then is to find the best way to reconstruct $x(t)$ from the given samples, and the key questions are: (i) What classes of signals can be reconstructed? (ii) What classes of kernels allow such reconstructions? (iii) What kind of reconstruction algorithms are involved? Ideally, we would like to be able to reconstruct large classes of signals, using simple reconstruction algorithms and, most important, with general and physically realizable kernels.

This paper includes research conducted jointly with Pancham Shukla [6], Jesse Berent [2] and Loic Baboulaz [1].

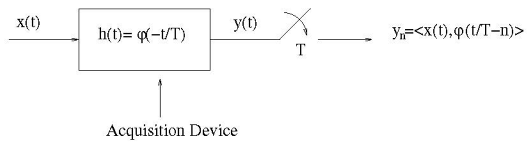

Fig. 1. Sampling setup. Here, $x(t)$ is the continous-time signal, $h(t)$ the impulse response of the acquisition device and T the sampling period. The measured samples are $y_n = \langle x(t), \varphi(t/T - n) \rangle$.

The classical answer to the sampling problem is provided by the famous Shannon sampling theorem which states the conditions to reconstruct bandlimited signals from their samples. In this case, the reconstruction process is linear and the kernel is the sinc function. Recently, it was shown that it is possible to develop sampling schemes for classes of signals that are neither bandlimited nor belong to a fixed subspace [11]. The common feature of such signals is that they have a parametric representation with a finite number of degrees of freedom and are, therefore, called signals with finite rate of innovation (FRI) [11].

In this paper, we further extend these results and show that many 1-D and 2-D signals with a local finite rate of innovation can be sampled and perfectly reconstructed using a wide range of sampling kernels and a local reconstruction algorithm. As in [11], the reconstruction process is based on the use of a locator or annihilating filter, a tool widely used in spectral estimation [7] and error correction coding [3]. In our context, the main property the kernel has to satisfy is to be able to reproduce polynomials or exponentials. Thus, functions satisfying Strang-Fix conditions (e.g., splines and scaling functions), exponential splines and functions with rational Fourier transforms can be used in our formulation. This last family of kernels is of particular importance since most linear devices used in practice have a transfer function which is rational.

The paper is organized as follows: In the next section we present the families of sampling kernels that are used in our sampling schemes. Section 3 presents our main sampling results, in particular, we show how to sample and perfectly reconstruct streams of Diracs. In Section 4, we use the results of the previous section to show that piecewise sinusoidal signals can be sampled as well. We then move to the 2-D case in Section 5, where an application to image super-resolution is also presented and finally conclude in Section 6.

2. SAMPLING KERNELS

As mentioned in the introduction, the signal $x(t)$ is usually filtered before being sampled. The samples y_n are given by $y_n = \langle x(t), \varphi(t/T-n)\rangle$, where the sampling kernel $\varphi(t)$ is the time reversed version of the filter's impulse response. The impulse response of the filter depends on the physical properties of the acquisition devise and, in most cases, is specified a-priori and cannot be modified. It is therefore important to develop sampling schemes that do not require the use of very particular or even physically non-realizable filters. In our formulation we can use a wide range of different kernels. For the sake of clarity, we divide them into two families:

1. *Polynomial reproducing kernels*: Any kernel $\varphi(t)$ that together with its shifted versions can reproduce polynomials of maximum degree N. That is, any kernel that satisfies

$$\sum_n c_{m,n}\varphi(t-n) = t^m \qquad m = 0,1,...,N \tag{1}$$

for a proper choice of the coefficients $c_{m,n}$.

2. *Exponential reproducing kernels*: Any kernel $\varphi(t)$ that together with its shifted versions can reproduce complex exponentials of the form $e^{\alpha_m t}$ with $\alpha_m = \alpha_0 + m\lambda$ and $m = 0,1,...,N$. That is, any kernel satisfying

$$\sum_n c_{m,n}\varphi(t-n) = e^{\alpha_m t} \tag{2}$$

for a proper choice of the coefficients $c_{m,n}$.

In both cases, the choice of N depends on the local rate of innovation of the original signal $x(t)$ as will become clear later on.

The first family of kernels includes any function satisfying the so-called Strang-Fix conditions [8] and therefore any scaling function that generates a wavelet basis. The second one includes any composite function of the form $\varphi(t) * \beta_{\vec{\alpha}}(t)$ with $\beta_{\vec{\alpha}}(t) = \beta_{\alpha_0}(t) * \beta_{\alpha_1}(t) * ... * \beta_{\alpha_N}(t)$ and $\alpha_m = \alpha_0 + m\lambda$ for $m = 0,1,...,N$, and where $\beta_{\alpha_m}(t)$ is an exponential spline (E-splines) [10]. This second family of kernels is of interest also because one can show that many functions with rational Fourier transform can be converted into functions that reproduce exponentials and can therefore be used to sample FRI signals.

3. RECONSTRUCTION OF 1-D FRI SIGNALS

In this section, we assume that the sampling kernel is of compact support L, that is, $\varphi(t) = 0$ for $t \notin [-L/2, L/2]$ where L is an integer for simplicity. We show that it is possible to sample and perfectly reconstruct streams of Diracs using kernels that reproduce polynomials or exponentials. Extensions of these results to the case of stream of differentiated Diracs, piecewise polynomial signals and non-uniform splines are also possible but are omitted.

The key feature of the reconstruction scheme is that it allows to retrieve the polynomial or the exponential moments of the original signal $x(t)$ from its samples. Since signals such as stream of Diracs are completely specified by a finite number of moments, perfect reconstruction is possible.

More precisely, assume that the sampling kernel $\varphi(t)$ satisfies the Strang-Fix conditions [8], that is, a linear combination of shifted versions of $\varphi(t)$ can reproduce polynomials of maximum degree N (see Equation (1)). Consider a stream, $x(t)$, of K Diracs: $x(t) = \sum_{k=0}^{K-1} a_k\delta(t-t_k)$, $t \in \mathbb{R}$. Call y_n the observed samples, that is, $y_n = \langle x(t), \varphi(t-n)\rangle$ where, for simplicity, we have assumed $T = 1$. Call $\tau_m = \sum_n c_{m,n}y_n$, $m = 0,1,...,N$ the weighted sum of the observed samples, where the weights $c_{m,n}$ are those in Equation (1) that reproduce t^m. We have that

$$\begin{aligned}\tau_m &= \textstyle\sum_n c_{m,n}y_n \\ &\overset{(a)}{=} \langle x(t), \textstyle\sum_n c_{m,n}\varphi(t-n)\rangle \\ &\overset{(b)}{=} \int_{-\infty}^{\infty}\textstyle\sum_{k=0}^{K-1} a_k\delta(t-t_k)t^m dt \\ &= \textstyle\sum_{k=0}^{K-1} a_k t_k^m \qquad m = 0,1,...,N\end{aligned} \tag{3}$$

where (a) follows from the linearity of the inner product and (b) from the polynomial reproduction formula in (1). The integral in (b) represents precisely the m-th order moment of the original signal $x(t)$. Hence, proper linear combinations of the observed samples provide the first $N+1$ moments of the signal.

The discrete signal $\tau_m = \sum_{k=0}^{K-1} a_k t_k^m$ $m = 0,1,..,N$ is very often encountered in spectral estimation and in that context the parameter a_k and t_k of τ_m are retrieved using the annihilating filter method. This method operates as follows:

Call h_m $m = 0,1,...,K$ the filter with z-transform

$$H(z) = \sum_{m=0}^{K} h_m z^{-m} = \prod_{k=0}^{K-1}(1 - t_k z^{-1}). \tag{4}$$

That is, the roots of $H(z)$ correspond to the locations t_k. It clearly follows that

$$h_m * \tau_m = \sum_{i=0}^{K} h_i\tau_{m-i} = \sum_{i=0}^{K}\sum_{k=0}^{K-1} a_k h_i t_k^{m-i} = 0. \tag{5}$$

The filter h_m is thus called annihilating filter since it annihilates the observed signal τ_m. The zeros of this filter uniquely define the set of locations t_k since the locations are distinct. The filter coefficients h_m are found from the system of equations in (5). Since $h_0 = 1$, the identity in (5) leads to a Yule-Walker system of equations involving at least $2K$ consecutive values of τ_m and has, in this case, a unique solution since h_m is unique for the given signal. Given the filter coefficients h_m, the locations of the Diracs are the roots of the polynomial in (4). Notice that, since we need at least $2K$ consecutive values of τ_m to solve the Yule-Walker system, we need the sampling kernel to be able to reproduce polynomials of maximum degree $N \geq 2K-1$.

Given the locations $t_0, t_1, ..., t_k$, the weights a_k are obtained by solving, for instance, the first K consecutive equations in (3). These equations lead to a Vandermonde system which yields a unique solution for the weights a_k given that the t_k's are distinct.

Thus, a stream of K Diracs is uniquely determined from the samples $y_n = \langle x(t), \varphi(t/T-n)\rangle$, if the sampling kernel $\varphi(t)$ can reproduce polynomials of maximum degree $N \geq 2K-1$. The problem is that the reconstruction scheme becomes more and more complex and unstable when the number K of Diracs increases. It is therefore critical to see if we can take advantage of the locality of the sampling kernel to develop a sequential, local reconstruction algorithm. Intuitively, if we have groups of Diracs separated by empty intervals, then we should be able to separate these groups and reconstruct them sequentially. Indeed, it is possible to show that if there are at most K Diracs in an interval of size $2KL$, we are assured that there is a sufficient number of zero samples that separates two groups of K Diracs and so it is possible to apply the above algorithm sequentially on each group of K Diracs. We can thus summarize the discussion of this section as follows:

Theorem 1 *Assume a sampling kernel $\varphi(t)$ that can reproduce polynomials of maximum degree $N \geq 2K-1$ and of compact support L. An infinite-length stream of Diracs $x(t) = \sum_{n\in\mathbb{Z}} a_n\delta(t-t_n)$ is uniquely determined from the samples defined by $y_n = \langle x(t), \varphi(t/T-n)\rangle$ if there are at most K Diracs in an interval of size $2KLT$.*

The situation stays the same when the kernel is able to reproduce exponentials rather than polynomials. Assume, for instance, that our kernel is able to reproduce exponential of the form $e^{\alpha_m t}$ with $\alpha_m = \alpha_0 + m\lambda$ and $m = 0, 1, ..., N$. For instance, $\varphi(t)$ is an E-spline $\beta_{\vec{\alpha}}(t)$ with $\vec{\alpha} = (\alpha_0, \alpha_1, ..., \alpha_N)$ and $\alpha_m = \alpha_0 + m\lambda$ or a composite function $\varphi(t) * \beta_{\vec{\alpha}}(t)$. Consider again a stream of K Diracs $x(t) = \sum_{k=0}^{K-1} a_k\delta(t-t_k)$. The samples y_n are then given by $y_n = \langle x(t), \varphi(t-n)\rangle$ and, using Eq. (2), it follows that

$$\begin{aligned} s_m &= \textstyle\sum_n c_{m,n} y_n = \int_{-\infty}^{\infty} x(t) e^{\alpha_0 + m\lambda t} dt \\ &= \textstyle\sum_{k=0}^{K-1} a_k e^{\alpha_0 + m\lambda t_k} \qquad m = 0, 1, ..., N. \end{aligned}$$

This means that proper linear combinations of the samples y_n lead to a signal s_m of the form $s_m = \sum_{k=0}^{K-1} a_k e^{\alpha_0 + m\lambda t_k}$. The interesting point is that the annihilating filter method can also be used when the observed signal is of the form $s_m = \sum_{k=0}^{K-1} a_k e^{\alpha_m t}$ and $\alpha_m = \alpha_0 + m\lambda$ and, therefore, the locations and the amplitudes of the Diracs can be retrieved using the annihilating filter method as in the polynomial case.

4. SAMPLING PIECEWISE SINUSOIDAL SIGNALS

Consider a piecewise sinusoidal signal like, for instance, the one shown in Figure 2(a). Clearly, it is not possible to sample it using classical Shannon theory since the signal is not bandlimited. Despite the fact that the signal has a finite number of degrees of freedom, previous methods would not work either. This happens because the signal contains innovation in both the temporal and the spectral domains. Yet, a proper combination of the annihilating filter method and the sampling theory developed before leads to an exact sampling scheme for this case as well.

Assume the signal to reconstruct is the one shown in Figure 2. The scheme operates as follows (for more details we refer to [2]): we first recover the sinusoidal parameters with an annihilating filter. Call $y_n = \langle x(t), \varphi(t-n)\rangle$ the observed samples where $\varphi(t)$ is a generic kernel. The samples generated by the sinusoid and that are not influenced by the two discontinuities can be annihilated using the filter $H(z) = (1 - e^{j\omega_0}z^{-1})(1 - e^{-j\omega_0}z^{-1})$. The filter's coefficients can be found solving a Yule-Walker system similar to the one shown before and the knowledge of h_n allows us to retrieve all the parameters of the sinusoid. The next step is to retrieve t_0 and t_1. By applying the filter h_n on the samples we have that ([2])

$$z_n = y_n * h_n = \langle x_\delta(t), \varphi(t-n) * \beta_{\vec{\alpha}}(t-n)\rangle,$$

where the new signal $x_\delta(t)$ is a stream of differentiated Diracs located at t_0 and t_1 and $\beta_{\vec{\alpha}}$ is an E-spline. Therefore, if the original kernel $\varphi(t)$ can reproduce polynomials or proper exponentials, we can retrieve the locations t_0 and t_1 using the results of Section 3. We therefore retrieve the entire signal.

The same methodology can be used also when the signals contains more pieces and more sinusoids for piece.

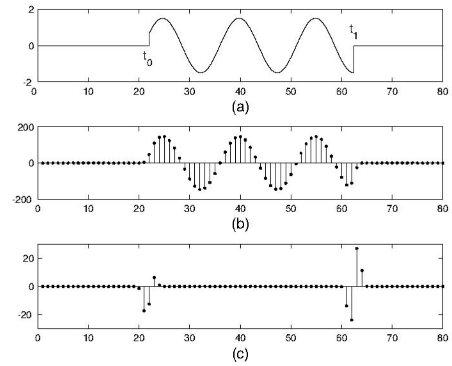

Fig. 2. Sampling piecewise sinusoidal signals. (a) Original piecewise sinusoidal signal $x(t)$ with one sinusoid of frequency ω_0. (b) Sampled signal. (c) Annihilated signal z_n. These samples are equivalent to those obtained by sampling a stream of differentiated Diracs located at t_0 and t_1 with the new kernel $\varphi(t) * \beta_{\vec{\alpha}}(t)$.

5. SAMPLING SCHEMES FOR 2-D SIGNALS WITH FINITE RATE OF INNOVATION

In this section we concentrate only on kernels that reproduce polynomials. In particular, we assume that the 2-D sampling kernel $\varphi_{xy}(x, y)$ is given by the tensor product of a 1-D function $\varphi(x)$ that reproduces polynomials. That is, $\varphi_{x,y}(xy) = \varphi(x)\varphi(y)$ and $\varphi(x)$ satisfies Eq (1).

The sampling schemes of Section 3 are based on the fact that a stream of K Diracs is uniquely determined by its first $2K$ moments. Since it is possible to retrieve these moments from the samples y_n, it is possible to reconstruct the original signal. The situation in 2-D is very similar, but complex rather than real moments are needed in this context.

Consider first a set of K 2-D Diracs. That is:

$$f(x, y) = \sum_{k=0}^{K} a_k \delta(x - x_k, y - y_k).$$

The samples are $y_{n,m} = \langle f(x, y), \varphi_{xy}(x-n, y-m)\rangle$ and, by construction, the kernel $\varphi_{xy}(x, y)$ is able to reproduce polynomials of the form $x^n y^l$, $n = 0, 1, ..., N$, $l = 0, 1, ..., N$. It is easy to show that with the right linear combination of the samples $y_{n,m}$, we can estimate the complex moments of $f(x, y)$ in much the same way as we estimated the real moments in the 1-D case. Thus, we end-up observing

$$\tau_m = \int\int f(x, y)(x + jy)^m dxdy \qquad m = 0, 1, ..., N.$$

Since $f(x, y)$ is a set of K Diracs, the complex moments of $f(x, y)$ have the following form

$$\tau_m = \sum_{k=0}^{K-1} a_k z_k^m$$

where the z_ks represent the locations of the K Diracs in complex form: $z_k = x_k + jy_k$. As in the 1-D case, the complex locations of the Diracs and their amplitudes are found using the annihilating filter

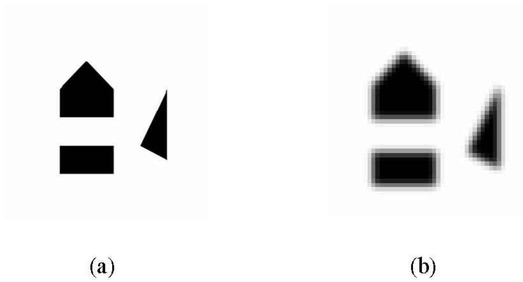

Fig. 3. (a) An original image $g(x, y)$ of size 3767×3767 pixels consists of three bilevel polygons: triangle, rectangle, and pentagon. (b) The set of 50×50 samples produced by the inner products of $g(x, y)$ with a B-Spline sampling kernel $\varphi_{xy}(x, y) = \beta^9_{xy}(x, y)$ with support 631×631 pixels that can reproduce polynomials up to degree nine. The original image is reconstructed from this samples exactly.

method. Therefore, as in the 1-D case, the reconstruction algorithm in 2-D operates in three steps:

1. Estimate the first $N \geq (2K - 1)$ complex moments τ_m of $f(x, y)$ from the samples $y_{n,m}$.
2. Find the filter h_m that annihilates τ_m. The roots of the filter represents the locations of the Diracs in complex form.
3. Estimate the amplitudes of the Diracs by solving a Vandermonde system.

If the kernel has compact support, it is possible to sample sets of Diracs with more than K Diracs. We just need group of at most K Diracs to be separated enough so that they can be reconstructed independently.

Bi-level polygonal images are also uniquely determined by their complex moments [4, 5]. Consider a simply connected convex polygon with K vertices, it is possible to show that [4, 5]

$$\hat{\tau}_m = m(m-1) \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x, y) z^{m-2} dx dy = \sum_{k=0}^{K-1} \rho_k z_k^m,$$

where the z_ks represent the locations of the vertices of the polygon in complex coordinates. Therefore, as in the previous case, by estimating the complex moments from the samples $y_{n,m}$ and by using the annihilating filter, we can retrieve the locations of the vertices of the polygons and, therefore, the original signal. An example of this sampling scheme is shown in Figure 3.

Moments are sometimes used for image registration. With the techniques presented above we are able to retrieve the original moments from undersampled images and use them for registration in the case we want to perform image super-resolution. An example of such an approach is shown in Figure 4, for more detail we refer to [1].

6. CONCLUSIONS

We have presented new schemes to sample signals with finite rate of innovation. We have shown that it is possible to sample and perfectly reconstruct many FRI signals using a wide range of sampling kernels. The classes of kernels that can be used include functions satisfying Strang-Fix conditions and therefore scaling functions for wavelet bases and E-splines.

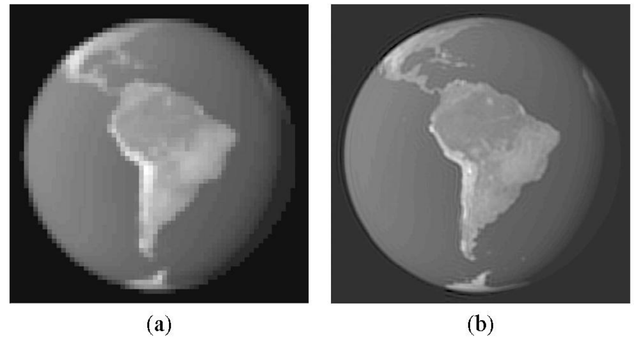

Fig. 4. Super-resolution using 100 images. (a) Low resolution image (65×65). (b) Super-resolved image (2000×2000)

Extensions to the 2-dimensional case have been presented as well and we have shown that these results find application in image super-resolution.

The cases of noisy measurements and of model mismatch are under investigation.

7. REFERENCES

[1] L. Baboulaz and P.L. Dragotti. Continuous moments from samples - new algorithms for distributed acquisition and image super-resolution. In *Proc. of IEEE Int. Conf. on Image Processing(ICIP) (submitted)*, Atlanta (GA), October 2006.

[2] J. Berent and P.L. Dragotti. Perfect reconstruction schemes for sampling piecewise sinusoisal signals. In *Proc. of IEEE Int. Conf. on acoustic, speech and signal processing (to appear)*, Toulouse (France), May 2006.

[3] R.E. Blahut. *Theory and Practice of Error Control Codes*. Addison-Wesley, 1983.

[4] P. J. Davis. Triangle formulas in the complex plane. *Math. Comput.*, 18:569–577, 1964.

[5] P. Milanfar, G. C. Verghese, W.C. Karl, and A. S. Willsky. Reconstructing polygons from moments with connections to array processing. *IEEE Trans. Signal Processing*, 43(2):432–443, February 1995.

[6] P. Shukla and P.L. Dragotti. Sampling schemes for 2-d signals with finite rate of innovation using kernels that reproduce polynomials. In *Proc. of IEEE Int. Conf. on Image Processing(ICIP)*, Genova (Italy), September 2005.

[7] P. Stoica and R Moses. *Introduction to Spectral Analysis*. Englewood Cliffs,NJ, Prentice-Hall, 2000.

[8] G. Strang and Fix. G. A Fourier analysis of the finite element variational method. In *Constructive Aspect of Functional Analysis*, pages 796–830, Rome, Italy, 1971.

[9] M. Unser. Sampling-50 years after Shannon. *Proc. IEEE*, 88:569–587, April 2000.

[10] M. Unser and T. Blu. Cardinal Exponential Splines: Part I-theory and filtering algorithms. *IEEE Trans. on Signal Processing*, 53(4):1425–1438, April 2005.

[11] M. Vetterli, P. Marziliano, and T. Blu. Sampling signals with finite rate of innovation. *IEEE Trans. Signal Processing*, 50(6):1417–1428, June 2002.